HIGHER ORDER (n-Order) ODE

SUMMARY

- Basically, the solution techniques for homogenous and non-homogenous n-order ODE are the same as for 2nd order ODE. We extend 2nd order ODE solution for n-order ODE.

- Since finding roots of polynomial order more than 3 is not easy. We confine our subject up to 3-rd order ODE problems only.

Homogenous n-Order DE

We now pay attention to the higher order DE of arbitrary order *n*,

Where $$y^{(n)} = \frac{d^n y}{dt^n}$$

Let's see now the case of linear homogenous DE with constant coefficient:

$$a_n y^{(n)} + a_{n-1}(t) y^{(n-1)} + \ldots + a_1(t) y' + a_0(t) y = 0$$

As in the case of 2nd order DE, then we need to check the roots of the polynomial characteristic equation (auxiliary equation=A.E).

Then there are basically the three cases: real distinct roots, repeated roots and complex roots. But, these are not always the case since the roots may be complex roots and repeated for instance.

So, for each n^{th} order differential equation we'll need to form a set of *n* linearly independent functions (*i.e.* a fundamental set of solutions) in order to get a general solution.

1. Real distinct roots

If there are *k* distinct roots of A.E as: $m_1, m_2, \dots, m_k$. Then the solution will have the form:

$$c_1 e^{m_1 t} + c_2 e^{m_2 t} + \dots + c_k e^{m_k t}$$

2. Real repeated roots

Let's suppose that *m* is a root of multiplicity *k* (*i.e.* *m* occurs *k* times in auxiliary/characteristic equation). The form of the solution here is a natural extension of the work we get in the 2nd order case:

$$c_1 e^{mt} + c_2 t e^{mt} + c_3 t^2 e^{mt} + \dots + c_k t^{k-1} e^{mt}$$

3. Complex roots

If the complex roots: **$\alpha \pm j\omega$** occurs only once. In this case we'll get the standard solution like the 2nd order case:

$$c_1 e^{\alpha t}\cos(\omega t) + c_2 e^{\alpha t}\sin(\omega t)$$

If the complex roots: **$\alpha \pm j\omega$** has multiplicity *k* (occurs *k* times). In this case we'll get the standard solution:

$$\begin{aligned}
&c_{11} e^{\alpha t}\cos(\omega t) + c_{12} e^{\alpha t}\sin(\omega t) + \\
&c_{21} t e^{\alpha t}\cos(\omega t) + c_{22} t e^{\alpha t}\sin(\omega t) + \ldots. \\
&\ldots. + c_{k1} t^{k-1} e^{\alpha t}\cos(\omega t) + c_{k2} t^{k-1} e^{\alpha t}\sin(\omega t)
\end{aligned}$$

Example 1 Solve the following IVP.

$$y^{(3)}-5y''-22y'+56y=0 \qquad y(0)=1 \quad y'(0)=-2 \quad y''(0)=-4$$

Solution

The characteristic equation is,

$$r^3-5r^2-22r+56=(r+4)(r-2)(r-7)=0 \qquad \Rightarrow \qquad r_1=-4,\ r_2=2,\ r_3=7$$

So we have three real distinct roots here and so the general solution is,

$$y(t)=c_1\mathbf{e}^{-4t}+c_2\mathbf{e}^{2t}+c_3\mathbf{e}^{7t}$$

Differentiating a couple of times and applying the initial conditions gives the following system of equations that we'll need to solve in order to find the coefficients.

$$\begin{aligned} 1&=y(0)=c_1+c_2+c_3 \\ -2&=y'(0)=-4c_1+2c_2+7c_3 \\ -4&=y''(0)=16c_1+4c_2+49c_3 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} c_1&=\tfrac{14}{33} \\ c_2&=\tfrac{13}{15} \\ c_3&=-\tfrac{16}{55} \end{aligned}$$

The actual solution is then,

$$y(t)=\tfrac{14}{33}\mathbf{e}^{-4t}+\tfrac{13}{15}\mathbf{e}^{2t}-\tfrac{16}{55}\mathbf{e}^{7t}$$

Example 2 Solve the following differential equation.

$$2y^{(4)} + 11y^{(3)} + 18y'' + 4y' - 8y = 0$$

Solution

The characteristic equation is,

$$2r^4 + 11r^3 + 18r^2 + 4r - 8 = (2r-1)(r+2)^3 = 0$$

So, we have two roots here, $r_1 = \frac{1}{2}$ and $r_2 = -2$ which is multiplicity of 3.

The general solution is then,

$$y(t) = c_1 \mathbf{e}^{\frac{1}{2}t} + c_2 \mathbf{e}^{-2t} + c_3 t \mathbf{e}^{-2t} + c_4 t^2 \mathbf{e}^{-2t}$$

Example 3: Find general solution for:

$$y^{(3)} + 4\dot{y} = 0$$

Auxiliary equation: $$r^3 + 4r = (r^2 + 4)r = 0$$

→r_1=0 (real distinct), $r_{2,3}$=±j2 (complex conjugate)
→General solution to the DE:

$$y(t) = A + B\cos 2t + C\sin 2t$$ #

Non-Homogenous n-Order DE

The linear non-homogenous nth order DE has the general form as:

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y' + p_0(t)y = r(t) \qquad \ldots(1)$$

For corresponding homogenous DE, then we have:

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y' + p_0(t)y = 0 \qquad \ldots(2)$$

Let's suppose that $y_1(t), y_2(t), \ldots\ldots, y_n(t)$ are a fundamental set of the solutions to (2).

Now, just as we did with the 2nd order, the general solution to (1) has the form:

$$y(t) = c_1y_1(t) + c_2y_2(t) + \ldots + c_ny_n(t) + Y_p(t) = y_c(t) + Y_p(t)$$

$y_c(t) = c_1y_1(t) + c_2y_2(t) + \ldots + c_ny_n(t)$ is complementary solution.

$Y_p(t)$ is the particular solution.

Non-Homogenous n-Order DE

Like in the case of 2nd order DE, we have two methods to find the solutions: Undetermined Coefficients and Variation of Parameters.

The method of undetermined coefficient is identical to what we have in the 2nd order DE.

Thus, we will only discuss:

Variation of Parameters (Wronskian).

Solution by Variation of Parameters

A particular solution to (1) is then given by:

$$Y_p(t) = y_1(t)\int \frac{r(t)W_1(t)}{W(t)}dt + y_2(t)\int \frac{r(t)W_2(t)}{W(t)}dt + \dots$$

$$\dots + y_n(t)\int \frac{r(t)W_n(t)}{W(t)}d$$

$$\Rightarrow Y_p(t) = y_1u_1 + y_2u_2 + \dots + y_nu_n$$

$$u_1 = \int \frac{r(t)W_1(t)}{W(t)}dt, \quad u_2 = \int \frac{r(t)W_2(t)}{W(t)}dt, \quad \cdots, \quad u_n = \int \frac{r(t)W_n(t)}{W(t)}dt$$

Solution by Variation of Parameters

Wronskian *W is*:

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = W\left(y_1, y_2, \ldots y_n\right)$$

And W_i we get by replacing i^{th} column of W with the column $[0,0,\ldots,1]^T$.

For example for n=2 (2nd order):

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix}$$

Therefore, this also works for 2nd order system.

$$Y_p(t) = y_1(t)\int \frac{r(t)W_1(t)}{W(t)}dt + y_2(t)\int \frac{r(t)W_2(t)}{W(t)}dt + \ldots$$

$$\ldots + y_n(t)\int \frac{r(t)W_n(t)}{W(t)}d$$

For 2nd order DE: $W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2$ $\quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1$

$$\longrightarrow \quad Y_p(t) = -y_1(t)\int \frac{r(t)y_2(t)}{W(t)}dt + y_2(t)\int \frac{r(t)y_1(t)}{W(t)}dt$$

Same as what we have in 2nd order DE.

Example 1 Solve the following differential equation.

$$y^{(3)} - 2y'' - 21y' - 18y = 3 + 4\mathbf{e}^{-t}$$

Solution

The characteristic equation is,

$$r^3 - 2r^2 - 21r - 18 = (r+3)(r+1)(r-6) = 0 \qquad \rightarrow \qquad r_1 = -3,\ r_2 = -1,\ r_3 = 6$$

So we have three real distinct roots here and so the general solution is,

$$y_c(t) = c_1\mathbf{e}^{-3t} + c_2\mathbf{e}^{-t} + c_3\mathbf{e}^{6t}$$

Okay, we've now got several determinants to compute. We'll leave it to you to verify the following determinant computations.

$$W = \begin{vmatrix} \mathbf{e}^{-3t} & \mathbf{e}^{-t} & \mathbf{e}^{6t} \\ -3\mathbf{e}^{-3t} & -\mathbf{e}^{-t} & 6\mathbf{e}^{6t} \\ 9\mathbf{e}^{-3t} & \mathbf{e}^{-t} & 36\mathbf{e}^{6t} \end{vmatrix} = 126\mathbf{e}^{2t} \qquad W_1 = \begin{vmatrix} 0 & \mathbf{e}^{-t} & \mathbf{e}^{6t} \\ 0 & -\mathbf{e}^{-t} & 6\mathbf{e}^{6t} \\ 1 & \mathbf{e}^{-t} & 36\mathbf{e}^{6t} \end{vmatrix} = 7\mathbf{e}^{5t}$$

$$W_2 = \begin{vmatrix} \mathbf{e}^{-3t} & 0 & \mathbf{e}^{6t} \\ -3\mathbf{e}^{-3t} & 0 & 6\mathbf{e}^{6t} \\ 9\mathbf{e}^{-3t} & 1 & 36\mathbf{e}^{6t} \end{vmatrix} = -9\mathbf{e}^{3t} \qquad W_3 = \begin{vmatrix} \mathbf{e}^{-3t} & \mathbf{e}^{-t} & 0 \\ -3\mathbf{e}^{-3t} & -\mathbf{e}^{-t} & 0 \\ 9\mathbf{e}^{-3t} & \mathbf{e}^{-t} & 1 \end{vmatrix} = 2\mathbf{e}^{-4t}$$

Example 1…cont'd

Now, given that $r(t) = 3 + 4\mathbf{e}^{-t}$ we can compute each of the u_i. Here are those integrals.

$$u_1 = \int \frac{\left(3+4\mathbf{e}^{-t}\right)\left(7\mathbf{e}^{5t}\right)}{126\mathbf{e}^{2t}}\,dt = \frac{1}{18}\int 3\mathbf{e}^{3t} + 4\mathbf{e}^{2t}\,dt = \frac{1}{18}\left(\mathbf{e}^{3t} + 2\mathbf{e}^{2t}\right)$$

$$u_2 = \int \frac{\left(3+4\mathbf{e}^{-t}\right)\left(-9\mathbf{e}^{3t}\right)}{126\mathbf{e}^{2t}}\,dt = -\frac{1}{14}\int 3\mathbf{e}^{t} + 4\,dt = -\frac{1}{14}\left(3\mathbf{e}^{t} + 4t\right)$$

$$u_3 = \int \frac{\left(3+4\mathbf{e}^{-t}\right)\left(2\mathbf{e}^{-4t}\right)}{126\mathbf{e}^{2t}}\,dt = \frac{1}{63}\int 3\mathbf{e}^{-6t} + 4\mathbf{e}^{-7t}\,dt = \frac{1}{63}\left(-\frac{1}{2}\mathbf{e}^{-6t} - \frac{4}{7}\mathbf{e}^{-7t}\right)$$

Finally, a particular solution for this differential equation is then,

$$\begin{aligned} Y_P &= u_1 y_1 + u_2 y_2 + u_3 y_3 \\ &= \tfrac{1}{18}\left(\mathbf{e}^{3t} + 2\mathbf{e}^{2t}\right)\mathbf{e}^{-3t} - \tfrac{1}{14}\left(3\mathbf{e}^{t} + 4t\right)\mathbf{e}^{-t} + \tfrac{1}{63}\left(-\tfrac{1}{2}\mathbf{e}^{-6t} - \tfrac{4}{7}\mathbf{e}^{-7t}\right)\mathbf{e}^{6t} \\ &= -\tfrac{1}{6} + \tfrac{5}{49}\mathbf{e}^{-t} - \tfrac{2}{7}t\mathbf{e}^{-t} \end{aligned}$$

The general solution is then,

$$y(t) = c_1\mathbf{e}^{-3t} + c_2\mathbf{e}^{-t} + c_3\mathbf{e}^{6t} - \tfrac{1}{6} + \tfrac{5}{49}\mathbf{e}^{-t} - \tfrac{2}{7}t\mathbf{e}^{-t}$$

Exercise:

1. Obtain the general solution of the following 3rd order homogenous differential equation:

$$y^{(3)} + 3y'' - 4y = 0$$

2. Obtain the general solution of the following non-homogenous differential equation using undetermined coefficient (guessing) and parameter variations (Wronskian):

$$y'' - 4y = 8e^{2x}$$

Thank you
To be continue…